Dominant-Strategy versus Bayesian Multi-item Auctions: Maximum Revenue Determination and Comparison

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We address two related unanswered questions in maximum revenue multi-item auctions. Is dominant-strategy implementation equivalent to the semantically less stringent Bayesian one (as in the case of Myerson's 1-item auction)? Can one find explicit solutions for non-trivial families of multi-item auctions (as in the 1-item case)? In this paper, we present such natural families whose explicit solutions exhibit a revenue gap between the two implementations. More precisely, consider the k-item n-buyer maximum revenue auction where $k, n > 1$ with additive valuation in the independent setting (i.e., the buyers i have independent private distributions F_{ij} on items j). We derive exact formulas for the maximum revenue when $k = 2$ and F_{ij} are any IID distributions on support of size 2, for both the dominant-strategy (DIC) and the Bayesian (BIC) implementations. The formulas lead to the simple characterization that, the two models have identical maximum revenue if and only if selling-separately is optimal for the distribution. Our results also give the first demonstration, in this setting, of revenue gaps between the two models. For instance, if $k = n = 2$ and $Pr\{X_F = 1\} = Pr\{X_F = 2\} = \frac{1}{2}$, then the maximum revenue in the Bayesian implementation exceeds that in the dominant-strategy by exactly 2%; the same gap exists for the continuous uniform distribution X_F over $[a, a + 1] \cup [2a, 2a + 1]$ for all large a.

Additional Key Words and Phrases: maximum revenue; multi-item auction; dominant strategy; Bayesian implementation

1 INTRODUCTION

We consider the k-item n-buyer maximum revenue auction with additive[1] valuation in the independent setting (i.e., the buyers i have independent private distributions F_i^j over the range $[0, \infty)$ on items j). How should the optimal mechanisms be designed?

Myersons's classical paper [24] elegantly solved the problem for the single item case. For multiple items ($k > 1$), the problem is much more complex with an extensive literature (see *Related Work* below). Much progress has been made, but many interesting questions remain open. In this paper we focus on two such questions that arise naturally.

Firstly, there are two standard models of mechanism design for auctions, known respectively as *dominant-strategy incentive-compatible* (DIC) and *Bayesian incentive-compatible* (BIC) mechanisms. Formally, the BIC constraints look much weaker than the DIC constraints. It is thus a remarkable feature of Myersons's theory that exactly the same maximum revenue is achieved by the BIC mechanisms and the DIC mechanisms for single-item auctions. Can this equivalence hold for $k > 1$?

Q1. For $k > 1$, can Bayesian incentive-compatible (BIC) mechanisms ever produce strictly more revenue than the dominant-strategy incentive-compatible (DIC) mechanisms?

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[1]Namely, for each buyer i, the valuation of a set S of items is the sum of valuations for all the items in S.

(A substantial literature exists on the above DIC versus BIC question: see *Related Work* below). For our second question, note that Myerson's characterization of optimal mechanisms for $k = 1$ leads to explicit formulas for the maximum revenue. For $k > 1$, in the single buyer ($n = 1$) case, there is a rich collection of sophisticated results (e.g., [14–19, 23, 25, 28]), where explicit expressions for optimal revenue are obtained for certain discrete and continuous distributions. However, for $k > 1$ and $n > 1$, there do not seem to be any interesting results of this kind in the literature. Thus we pose the following question:

Q2. For $k > 1$ and $n > 1$, can we obtain explicit expressions of the optimal revenue for interesting families of distributions?

In this paper we address questions Q1 and Q2. In the direction of Q2, we derive exact formulas for the maximum revenue for both DIC and BIC implementations for $k = 2$ and any $n > 1$, where the 2n distributions F_i^j are IID with a common F of support size 2. As a by-product, these formulas give an answer to Q1, showing the BIC optimal revenue expression to be strictly greater than that of DIC for a broad range of parameters. In fact the formulas lead to the simple characterization that, the two implementations have identical maximum revenue if and only if selling-separately is optimal for the distribution. For instance, if $k = n = 2$ and $Pr\{X_F = 1\} = Pr\{X_F = 2\} = \frac{1}{2}$, then the maximum revenue in the Bayesian implementation exceeds that in the dominant-strategy by exactly 2%. A natural extension to the continuous case shows that the same 2% gap holds for the uniform distribution over $[a, a + 1] \cup [2a, 2a + 1]$ as $a \to \infty$. We also remark that our result complements a theorem in [30] that the BIC maximum revenue is always upper bounded by a constant factor of the DIC maximum revenue.

Beyond providing an answer to Q1 and Q2, our techniques may have several other contributions. Firstly, it is demonstrated in a natural context how to turn a DIC mechanism into a BIC mechanism with increased revenue (see a specific example in Section 3.3). Secondly, our proposed optimal mechanisms, while applicable to arbitrary n buyers, have simple descriptions. Each mechanism employs only a pure hierarchical allocation rule. Such simple designs may lend these mechanisms to other applications. Finally, the problem of finding explicit exact solutions to multi-item auctions is an interesting open area. Economic concepts and interpretations often value precision over constant approximations. A collection of exactly solvable auction problems could be valuable for other econometrics explorations.

The main results of this paper will be stated in Section 3 with proofs given in Sections 4 and 5. Formal descriptions of our mechanisms and formulas will be illustrated through concrete examples (see Example 1 in Section 3.1 and 3.3) to help the understanding.

Related Work

Regarding Q1, when the independence condition on the distributions F_i^j is dropped, then the answer is known. [9] showed BIC can generate unbounded more revenue than DIC, when F_i^j are *correlated across buyers* even for $k = 1$. Recently, [29] showed in some instance with $k > 1$, BIC can generate strictly more revenue than DIC, when F_i^j are *correlated across items*. There are other examples (e.g. [13, 23]) where DIC and BIC are shown to be inequivalent in revenue (and other attributes), but their models are farther away from our model under consideration here.

We note that much progress has been made on the computational aspects of multi-item auctions. The intrinsic complexity of computing the optimal revenue has been investigated (e.g. [8, 10]; efficient algorithms have been found in a variety of circumstances (e.g. [4, 5, 11]); furthermore, simple approximation mechanisms have been extensively studied in various environments (e.g. [1, 6, 7, 12, 18, 21, 22, 26, 27, 30]).

2 PRELIMINARIES

2.1 Basic Concepts

Let $\mathcal{F}$ be a multi-dimensional distribution on $[0, \infty)^{nk}$. Consider the k-item n-buyer auction problem where the valuation $n \times k$ matrix $t = (t_i^j)$ is drawn from $\mathcal{F}$. Each buyer i has $t_i \equiv (t_i^1, t_i^2, \cdots, t_i^k)$ as his valuations of the k items. We also refer to t_i as buyer i's *type*, and t as the *type profile* of the buyers (or *profile* for short). For convenience, let t_{-i} denote the valuations of all buyers except buyer i; that is, $t_{-i} = (t_{i'} \mid i' \neq i)$. Note that t^j, the j-th column of the matrix t, contains the valuations of all the buyers on item j.

A *mechanism* M specifies an *allocation* $q(t) = (q_i^j(t)) \in [0, \infty)^{nk}$, where $q_i^j(t)$ denotes the probability that item j is allocated to buyer i when $t = (t_i^j)$ is reported as the type profile to M by the buyers. We require that $\sum_{i=1}^n q_i^j(t) \leq 1$ for all j, so that the total probability of allocating item j is at most 1. M also specifies a *payment* $s_i(t) \in (-\infty, \infty)$ for buyer i.

The *utility* $u_i(t)$ for buyer i is defined to be $t_i \cdot q_i(t) - s_i(t)$, where $t_i \cdot q_i(t)$ stands for the inner product $\sum_{j=1}^k t_i^j q_i^j(t)$. Let $u_i(t_i \leftarrow t_i', t_{-i}) = t_i \cdot q_i(t_i', t_{-i}) - s_i(t_i', t_{-i})$, i.e. the utility buyer i would obtain if he has type t_i but reports to the seller as t_i'. The *expected allocation* $\bar{q}_i(t_i)$ for buyer i is defined to be $E_{t_{-i}}(q(t_i, t_{-i}))$. The *expected utility* $\bar{u}_i(t_i)$ for buyer i is defined to be $E_{t_{-i}}(u(t_i, t_{-i}))$. Also let $\bar{u}_i(t_i \leftarrow t_i') = E_{t_{-i}}(\bar{u}_i(t_i \leftarrow t_i', t_{-i}))$.

The following formulas are well known:

> *Transfer Equations:* $u_i(t_i \leftarrow t_i', t_{-i}) = u_i(t_i', t_{-i}) + (t_i - t_i') \cdot q_i(t_i', t_{-i})$ for all t_i, t_i', t_{-i}.
>
> *Averaged Form:* $\bar{u}_i(t_i \leftarrow t_i') = \bar{u}_i(t_i') + (t_i - t_i') \cdot \bar{q}_i(t_i')$ for all t_i, t_i'.

Two kinds of mechanisms have been widely studied, referred to as *Dominant-strategy* and *Bayesian* implementations as specified below.

> **Dominant-strategy:** *IR* conditions: $u_i(t) \geq 0$ for all i and t.
> *DIC* conditions: $u_i(t_i, t_{-i}) \geq u_i(t_i \leftarrow t_i', t_{-i})$ for all t_i, t_i' and t_{-i}, or equivalently,
> *DIC* conditions (alternate): $u_i(t_i, t_{-i}) - u_i(t_i', t_{-i}) \geq (t_i - t_i') \cdot q_i(t_i', t_{-i})$ for all t_i, t_i' and t_{-i}.
> A mechanism is called *individually rational (IR)/dominant-strategy incentive compatible (DIC)*, if it satisfies the IR conditions/the DIC conditions, respectively.
>
> **Bayesian:** *BIR* conditions: $\bar{u}_i(t_i) \geq 0$ for all i and t_i.
> *BIC* conditions: $\bar{u}_i(t_i) \geq \bar{u}_i(t_i \leftarrow t_i')$ for all t_i, t_i', or equivalently,
> *BIC* conditions (alternate): $\bar{u}_i(t_i) - \bar{u}_i(t_i') \geq (t_i - t_i') \cdot \bar{q}_i(t_i')$ for all t_i, t_i'.
> A mechanism is called *Bayesian individually rational (BIR)/Bayesian incentive compatible (BIC)*, if it satisfies the BIR conditions/the BIC conditions, respectively.

Let $s(x) = \sum_{i=1}^n s_i(x)$ be the total payments received by the seller. For any mechanism M on $\mathcal{F}$, let $M(\mathcal{F}) = E_{x \sim \mathcal{F}}(s(x))$ be the (expected) revenue received by the seller from all buyers. The *optimal revenue* is defined as $REV_D(\mathcal{F}) = \sup_M M(\mathcal{F})$ when M ranges over all the IR-DIC mechanisms. Similarly, in the Bayesian model, the optimal revenue is defined as $REV_B(\mathcal{F}) = \sup_M M(\mathcal{F})$ where M ranges over all the BIR-BIC mechanisms. As a benchmark for comparison, let $SREV(\mathcal{F})$ stand for the revenue yielded when each item is sold separately by using the optimal mechanism of [24].

2.2 Hierarchy Allocation

The optimal BIC and DIC mechanisms proposed in this paper for n-buyer 2-item auctions will be described using the formalism of *hierarchy allocations*. The concept of hierarchy allocation was first raised in [2, 3] for one item, and later for multi-items in [4]. Here we only need the concept as a convenient language to succinctly present our mechanisms.

Consider an n-buyer 1-item auction. A *hierarchy allocation scheme H* is specified by a mapping $Rank : T \to \mathcal{R} \cup \{\infty\}$. Given a type $t \in (t_1, t_2, \ldots, t_n)$, scheme H allocates the item uniformly among the set of buyers i with the smallest ranking. If $Rank(t_i) = \infty$ for all i, then no allocation will be made to any buyer. For convenience, we use the notation $H = [\tau_{11}, \ldots, \tau_{1a_1}; \tau_{21}, \ldots, \tau_{2a_2}; \ldots; \tau_{\ell 1}, \ldots, \tau_{\ell a_\ell}]$ with the understanding $Rank(\tau_{dm}) = d$ for all $1 \leq d \leq \ell$, $1 \leq m \leq a_d$, and $Rank(t) = \infty$ for any type t not listed among τ_{dm}.

In an n-buyer k-item auction, a *hierarchy mechanism M* uses a *hierarchy allocation function* specified by a k-tuple $\mathcal{H} = (H^1, H^2, \cdots, H^k)$, where each H^j is a hierarchy allocation scheme to be used for item j; also a utility function $u_i(t)$ for each buyer i needs to be specified for M. Note that the payment for buyer i is determined by $s_i(t) = \sum_j q_i^j(t)t_i^j - u_i(t)$.

3 MAIN RESULTS

In this paper, we solve for $REV_D(\mathcal{F})$ and $REV_B(\mathcal{F})$ in the n-buyer, 2-item case when $\mathcal{F}$ consists of 2n IID's of a common F with support size 2. Any such $\mathcal{F}$ can be specified by a 4-tuple $\delta = (n, p, a, b)$ where $n \geq 2$ is an integer, $0 < p < 1$, and $0 \leq a < b$. Let $\mathcal{F}_\delta$ denote the valuation distribution for the n-buyer 2-item auction, where the distributions F_i^j for buyer i and item j are independent and identical (IID) copies of random variables X defined by $Pr\{X = a\} = p$ and $Pr\{X = b\} = 1 - p$. Assuming additive valuation on items for each buyer, we are interested in determining $REV_D(\mathcal{F}_\delta)$ and $REV_B(\mathcal{F}_\delta)$, the maximum revenue achievable under IR-DIC and BIR-BIC, respectively, for distribution $\mathcal{F}_\delta$. We find two benchmarks relevant, $SREV(\mathcal{F}_\delta)$ and $s_b = 2(1 - p^n)b$: the former is the revenue obtained by selling separately each item using Myerson's optimal mechanism; the latter is the revenue by selling separately each item at price b.

Fact 1. $SREV(\mathcal{F}_\delta) = 2 \cdot \max\{(1 - p^n)b, \ p^{n-1}a + (1 - p^{n-1})b\}$.

It suffices to show that, for 1-item auction with n buyers and IID distributions $\mathcal{F}_\delta$, the maximum revenue possible is equal to $\max\{(1 - p^n)b, \ p^{n-1}a + (1 - p^{n-1})b\}$. According to Myerson's optimal auction theory, the revenue maximization problem in this setting reduces to the welfare maximization problem where the valuation is replaced by a modified valuation (dependent on the distribution), called the *ironed virtual valuation* ϕ. It is easy to show that for the distribution $\mathcal{F}_\delta$, $\phi(a) = (a - (1 - p)b)/p$ and $\phi(b) = b$. The maximum welfare achievable is given by $p^n \max\{\phi(a), 0\} + (1 - p^n)\phi(b)$, which is exactly $\max\{(1 - p^n)b, \ p^{n-1}a + (1 - p^{n-1})b\}$. This proves Fact 1.

3.1 The Main Theorem

For any real-valued function G, we use G_+ to denote the nonnegative function defined as $G_+ = \max\{G, 0\}$. The functions $r_D(\delta)$ and $r_B(\delta)$ below will be used to express revenues.

Definition 3.1. Let $p_0 = p^{2n}$, $p_1 = 2np^{2n-1}(1 - p)$, and $p_2 = 2p^n \left(1 - p^n - np^{n-1}(1 - p)\right)$. Define

$$r_D(\delta) = 2(1 - p^n)b + p_0\left[2a - \frac{1 - p^2}{p^2}(b - a)\right]_+ + p_1\left[a - \frac{1 - p}{2p}(b - a)\right]_+$$
$$+ p_2\left[a - \frac{1 - p}{p}(b - a)\right]_+$$

$$r_B(\delta) = 2(1 - p^n)b + p_0\left[2a - \frac{1 - p^2}{p^2}(b - a)\right]_+ + (p_1 + p_2)\left[a - \frac{1 - p}{2p}(b - a)\right]_+$$

THEOREM 3.2 (**MAIN THEOREM**). $REV_B(\mathcal{F}_\delta) = r_B(\delta)$ and $REV_D(\mathcal{F}_\delta) = r_D(\delta)$.
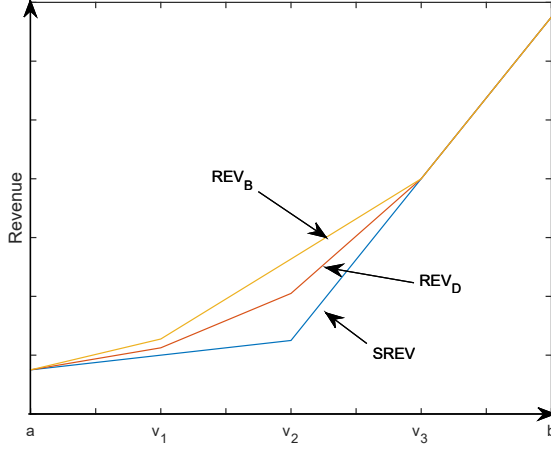
Fig. 1. REV_B, REV_D, and $SREV$ as functions of b

COROLLARY 3.3.

$$(a)\ REV_B(\mathcal{F}_\delta) = REV_D(\mathcal{F}_\delta) = SREV(\mathcal{F}_\delta)\ \ if\ b \geq a(1+p)/(1-p),$$
$$REV_B(\mathcal{F}_\delta) > REV_D(\mathcal{F}_\delta) > SREV(\mathcal{F}_\delta)\ \ otherwise.$$
$$(b)\ REV_D(\mathcal{F}_\delta) = REV_B(\mathcal{F}_\delta)\ \ if\ and\ only\ if\ Selling\text{-}Separately\ is\ optimal.$$

Remark 1. For fixed n, p, a, the functions $r_B(\delta)$, $r_D(\delta)$ and $SREV(\mathcal{F}_\delta)$ are each continuous, piecewise-linear functions in b as shown in Figure 1. Breakpoints of linearity occur at $v_1 = \frac{1+p^2}{1-p^2}a$, $v_2 = \frac{1}{1-p}a$, and $v_3 = \frac{1+p}{1-p}a$ along the b-axis. The last two additive terms of $r_B(\delta)$ are 0 when $b \geq v_1$, and $b \geq v_3$, respectively. Similarly, the last three additive terms of $r_D(\delta)$ are 0 when $b \geq v_1$, $b \geq v_3$, and $b \geq v_2$, respectively.

Remark 2. The formula for $r_D(\delta)$ can be interpreted as follows (and likewise for $r_B(\delta)$). The first term $2(1-p^n)b$ is equal to s_b. The three additive terms represent the extra revenue, beyond selling-separately at b, that can be gleaned from three specific subsets of profiles. (These subsets are defined as S_0, S_1, S_2 in Definition 5 later, with non-zero probability of occurrence p_0, p_1, p_2, respectively.)

Corollary 3.3 can be derived as follows. When $b \geq v_3 = \frac{1+p}{1-p}a$, all terms in square brackets in Definition 1 are 0, hence $r_D(\delta) = r_B(\delta) = 2(1-p^n)b = s_b$. As by definition $s_b \leq SREV(\mathcal{F}_\delta) \leq r_D(\delta)$, we conclude $r_D(\delta) = r_B(\delta) = SREV(\mathcal{F}_\delta)$ in this case. When $b < v_3$, we have $a - \frac{1-p}{2p}(b-a) > 0$, implying

$$r_B(\delta) - r_D(\delta) = p_2\left(\left[a - \frac{1-p}{2p}(b-a)\right]_+ - \left[a - \frac{1-p}{p}(b-a)\right]_+\right) > 0.$$

To compare $r_D(\delta)$ with $SREV(\mathcal{F}_\delta)$ when $b < v_3$, notice that the two continuous piecewise-linear functions are equal when $b = a$ and $b = v_3$. It is easy to check from their formulas that, $r_D(\delta)$ strictly dominates $SREV(\mathcal{F}_\delta)$ at both breakpoints v_1 and v_2 interior to $[a, v_3]$. We conclude that $r_D(\delta) > SREV(\mathcal{F}_\delta)$ over the entire interval (a, v_3). Corollary 3.3(a) follows. Corollary 3.3(b) follows immediately from (a). □

Example 1. As an illustration, consider the case $\delta = (n, p, a, b)$ with $n = 2$, $p = \frac{1}{2}$, $a = 1$, $b = 2$. The formulas in the Main Theorem tell us $REV_D(\mathcal{F}_\delta) = \frac{1}{8}(3 + 11b)$ for $b \in [2, 3)$, and $REV_B(\mathcal{F}_\delta) = \frac{1}{16}(9 + 21b)$ for $b \in [\frac{5}{3}, 3)$. Thus for $b = 2$ we have $REV_D(\mathcal{F}_\delta) = 25/8 = 3.125$ and $REV_B(\mathcal{F}_\delta) = 51/16 = 3.1875$, with a gap of 2%. Note that grand bundling yields only revenue $45/16$, while selling separately yields revenue 3, both strictly less than 3.125.

Open Questions: It is an interesting open question to determine whether the 2% gap in Example 1 is the largest gap between DIC and BIC revenue for the family of distributions considered in our model. It would also be interesting to extend our results to more general models, say, where the two items have distinct IID distributions.

3.2 Optimal Mechanisms

The optimal revenue $r_D(\delta)$ and $r_B(\delta)$ stated in the Main Theorem can be realized, respectively, by the IR-DIC mechanism $M_{D,\delta}$ and the IR-BIC mechanism $M_{B,\delta}$ defined below. First, we name the characteristic functions for the three intervals where the individual terms of $r_D(\delta)$, $r_B(\delta)$ are non-zero.

Definition 3.4. Define

$$\begin{cases} \alpha_{p,a,b} = 1 & \text{if } b < v_1, \text{ and } 0 \text{ otherwise.} \\ \beta_{p,a,b} = 1 & \text{if } b < v_3, \text{ and } 0 \text{ otherwise.} \\ \gamma_{p,a,b} = 1 & \text{if } b < v_2, \text{ and } 0 \text{ otherwise.} \end{cases}$$

The subscripts in $\alpha_{p,a,b}$, $\gamma_{p,a,b}$, $\beta_{p,a,b}$ can be dropped when p, a, b are clear from the context. Note that, if desired, the formulas for $r_D(\delta)$ and $r_B(\delta)$ can be written using α, β, γ as multipliers in place of the notation $G_+ \equiv \max\{G, 0\}$.

Definition 3.5. In what follows, the term *profile* refers to a profile in the support of $\mathcal{F}_\delta$, a *type* refers to a type in $\{a, b\} \times \{a, b\}$. For any profile t and $j \in \{1, 2\}$, we say t^j is *cheap* if $t_i^j = a$ for all $1 \leq i \leq n$ (we also say item j is cheap); otherwise t^j is *non-cheap*. Call a profile t *1-cheap* if t has exactly 1 cheap item. We use $I(t)$ to denote the subset of buyers i with $t_i \neq (a, a)$, that is, only excluding those who value both items at a. Note that, if t is 1-cheap, then $|I(t)|$ is equal to the number of b's in t (and all appearing in the same column).

We now define mechanism $M_{D,\delta}$ and $M_{B,\delta}$ below, using the language of hierarchy mechanism. First divide the range (a, ∞) of b into 4 subintervals: $I_1 = (a, v_1)$, $I_2 = [v_1, v_2)$, $I_3 = [v_2, v_3)$ and $I_4 = [v_3, \infty)$.

ALGORITHM 1: Mechanism $M_{D,\delta}$

Case 1. $b \in I_1$. Use the hierarchy allocation function (H^1, H^2) where $H^1 = [(b, b); (b, a); (a, b); (a, a)]$ and $H^2 = [(b, b); (a, b); (b, a); (a, a)]$;

Case 2. $b \in I_2$. Use the hierarchy allocation function (H^1, H^2) where $H^1 = [(b, b); (b, a); (a, b)]$ and $H^2 = [(b, b); (a, b); (b, a)]$;

Case 3. $b \in I_3$. If $t = (t_i, t_{-i})$ with t_{-i} being the lowest profile $(a, a)^{n-1}$, then offer items 1 and 2 to buyer i as a bundle at price $a + b$. Otherwise, use the hierarchy allocation function (H^1, H^2) where $H^1 = [(b, b); (b, a)]$, $H^2 = [(b, b); (a, b)]$;

Case 4. $b \in I_4$. Use the hierarchy allocation function (H^1, H^2) where $H^1 = [(b, b); (b, a)]$, $H^2 = [(b, b); (a, b)]$;

The payment of $M(\mathcal{F}_\delta)$ is determined by the following utility function: for $1 \le i \le n$, $t = (t_i, t_{-i})$,

$$u_i(t_i, t_{-i}) = \begin{cases} (b-a)\frac{\alpha}{n} & \text{if } t_i = (b,a) \text{ or } (a,b), \text{ and } t_{-i} = (a,a)^{n-1} \\ (b-a)(\frac{\alpha}{n} + \beta) & \text{if } t_i = (b,b), \text{ and } t_{-i} = (a,a)^{n-1} \\ (b-a)\frac{\gamma}{1+|I(t_{-i})|} & \text{if } t_i = (b,b), \ t_{-i} \text{ is 1-cheap} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Remark 3. Strictly speaking, $M_{D,\delta}$ in Case 3 is not a hierarchy mechanism. We abuse the term slightly for convenience. Observe that, when $b \in I_4$, $M_{D,\delta}$ can be described as selling each item separately at price b with a particular tie-breaking rule as dictated by the Case 4 allocation function (H^1, H^2). When $b \in I_3$, $M_{D,\delta}$ can be described as follows: if $t = (t_i, t_{-i})$ with $t_{-i} = (a,a)^{n-1}$, then offer items 1 and 2 to buyer i as a bundle at price $a + b$; otherwise sell each item separately at price b with a particular tie-breaking rule as dictated by the Case 3 allocation function (H^1, H^2).

ALGORITHM 2: Mechanism $M_{B,\delta}$

Case 1. $b \in I_1$: Use the hierarchy allocation function as defined in Case 1 of $M_{D,\delta}$;
Case 2. $b \in I_2 \cup I_3$: Use the hierarchy allocation function as defined in Case 2 of $M_{D,\delta}$;
Case 3. $b \in I_4$: Define $M_{B,\delta} = M_{D,\delta}$;

In both Case 1 and 2, the payment is defined by the same utility function $u_i(t_i, t_{-i})$ as in $M_{D,\delta}$ with only one *exception*: if $t_i = (b,b)$ and t_{-i} is 1-cheap, then let

$$u_i(t_i, t_{-i}) = \frac{1}{2}(b-a)\frac{\beta}{1 + |I(t_{-i})|}.$$

Remark 4. In fact, $M_{B,\delta}(\mathcal{F}_\delta)$ is an IR-BIC mechanisms (not just BIR-BIC), as will be shown later.

3.3 Example 1 Revisited

As an illustration, let us apply mechanisms $M_{D,\delta}$ and $M_{B,\delta}$ to the distribution $\delta = (n, p, a, b)$ in Example 1 where $n = 2$, $p = 1/2$, $a = 1$ and $b \in [2,3]$. Since both mechanisms are designed to be fully symmetric with respect to the swapping of items 1 and 2, and to the swapping of buyers 1 and 2, we only need to specify enough details up to these symmetries. This δ falls under Case 3 of mechanism $M_{D,\delta}$ and Case 2 of $M_{B,\delta}$, respectively.

ALGORITHM 3: Mechanism $M_{B,\delta}$ for Example 1

Case 1. If $t_2 = (1,1)$, then buyer 1 is offered both items as a bundle at price $1 + b$;
Case 2. If $t_2 = (1, b)$: ;
if $t_1 = (1, b)$, then each buyer is offered 50% of both items as a bundle at price $\frac{1}{2}(1 + b)$;
if $t_1 = (b, 1)$, then each buyer gets his most-valued item at price b;
if $t_1 = (b, b)$, then buyer 1 gets both items as a bundle at price $2b - \frac{1}{4}(b - 1)$;
Case 3. If $t_1 = t_2 = (b, b)$, then each buyer gets 50% of both items as a bundle at price b;

Question: What is the difference between the allocations by $M_{D,\delta}$ and $M_{B,\delta}$ in Example 1, and how does $M_{B,\delta}$ manage to outperform $M_{D,\delta}$?
Ans. The two mechanisms differ only in the way they handle the following two sets of profiles:
(A) Assume $t_1 = t_2 = (1, b)$ or $t_1 = t_2 = (b, 1)$. Here $M_{B,\delta}$ offers each buyer 50% of both items as a bundle at price $\frac{1}{2}(1 + b)$, while $M_{D,\delta}$ offers each buyer 50% of item 2 at price $\frac{b}{2}$;

ALGORITHM 4: Mechanism $M_{D,\delta}$ for Example 1

If any buyer i submits the bid $t_i = (1, 1)$, then the other buyer is offered items 1 and 2 as a bundle at price $1 + b$; otherwise, the items are sold separately at price b each (with a particular tie-breaking rule specified by the hierarchy allocation function (H^1, H^2) where $H^1 = [(b, b); (b, a)]$, $H^2 = [(b, b); (a, b)]$);

(B) Assume i has type (b, b) and the other buyer has type $(1, b)$ or $(b, 1)$. Here $M_{B,\delta}$ offers buyer i both items as a bundle at price $2b - \frac{1}{4}(b - 1)$, while $M_{D,\delta}$ offers buyer i both items as a bundle at price $2b$.

Mechanism $M_{B,\delta}$ gets more payment than $M_{D,\delta}$ in situation A and gets less in situation B, but gains an overall improvement of $\frac{1}{16}(3 - b)$ over $M_{B,\delta}$. It is key to observe that mechanism $M_{B,\delta}$ *violates* the DIC constraint $u_1((b, b), (1, b)) \geq u_1((1, b), (1, b)) + (b - 1)q_1^1((1, b), (1, b))$.

3.4 Application to Continuous Distributions

The results in the Main Theorem have implications on the maximum revenue for continuous distributions if the latter can be well approximated by $\mathcal{F}_\delta$. As an application, let $\lambda > 1$, $a > \frac{1}{\lambda - 1}$, and let $\mathcal{F} = (F_i^j \mid 1 \leq i \leq n, 1 \leq j \leq 2)$ be a distribution where $F_i^j = F$ are IID distributions with $support(X_F) = [a, a + 1] \cup [\lambda a, \lambda a + 1]$; let $p = Pr\{X_F \leq a + 1\}$. We can regard F_δ, where $\delta = (n, p, 1, \lambda)$, as a normalized discrete approximation of $\mathcal{F}$.

COROLLARY 3.6. *Let $\delta = (n, p, 1, \lambda)$. There exists a constant C_δ such that*

$$|REV_Z(\mathcal{F}) - r_Z(\delta) \cdot a| < C_\delta \ \text{for } Z \in \{D, B\}.$$

Corollary 3.6 is proved by an extension of our proof of the Main Theorem to the continuous setting (details omitted here).

Remark 5. There are general high-precision approximation theorems in the literature (e.g. see[6, 11, 20, 27]) connecting continuous and discrete distributions for the BIC maximum revenue auction. Our derivation of Corollary 3.6 does not rely on such general theorems.

We consider an illustrative example of Corollary 3.6 where $n = 2$. Let $\mathcal{G}_a = (F_i^j \mid i, j \in \{1, 2\})$, where $F_i^j = F$ are IID distributions with X_F uniformly distributed over $[a, a + 1] \cup [2a, 2a + 1]$. According to Corollary 3.6, we have $\frac{1}{a} \lim_{a \to \infty} REV_Z(\mathcal{G}_a) = r_Z(\delta)$ for $Z \in \{D, B\}$ where $\delta = (2, \frac{1}{2}, 1, 2)$. More precise bounds for this example are given below. Note that from Definition 1, one has $r_D(\delta) = \frac{25}{8}$ and $r_B(\delta) = \frac{51}{16}$, with a 2% difference.

COROLLARY 3.7. *For $a \geq 20$, the BIC maximum revenue for $\mathcal{G}_a$ strictly exceeds its DIC maximum revenue. In fact, we have for $a \geq 6$,*

$$\frac{25}{8}a \leq REV_D(\mathcal{G}_a) < \frac{25}{8}a + \frac{5}{4}$$

$$\frac{51}{16}a \leq REV_B(\mathcal{G}_a) < \frac{51}{16}a + \frac{3}{2}.$$

4 DIC MAXIMUM REVENUE

In this section we give a proof outline of the Main Theorem for the dominant strategy implementation:

THEOREM 4.1. *Any IR-DIC mechanisms M must satisfy $M(\mathcal{F}_\delta) \leq r_D(\delta)$.*

THEOREM 4.2. *$M_{D,\delta}$ is IR-DIC, and $M_{D,\delta}(\mathcal{F}_\delta) = r_D(\delta)$.*

We begin with a general discussion applicable to any mechanism. Let M be a mechanism with allocation q_i^j and utility u_i. We separate out the allocation of cheap items from non-cheap items. Thus, define $q'^j_i(t) = q_i^j(t)\eta^j(t)$ where $\eta^j(t) = 1$ if item j is cheap, and $\eta^j(t) = 0$ if j is non-cheap. Note that the welfare of the buyers from the allocation of cheap items is $a \cdot \sum_{i,j,t} Pr\{t\}q'^j_i(t)$, while the welfare from the non-cheap items is $\sum_{i,j,t} Pr\{t\}(1 - \eta^j(t))q_i^j(t)t_i^j$ which is at most $2(1 - p^n)b$ (the revenue obtained by selling-separately at price b). By definition of utility, it is clear that the revenue $M(\mathcal{F}_\delta)$ equals the total welfare minus utility of the buyers. This leads to the following formula:

Basic Formula. For any mechanism M, we have

$$M(\mathcal{F}_\delta) \le 2(1 - p^n)b + Qa - U,$$

where $Q = \sum_{i,j,t} Pr\{t\}q'^j_i(t)$ and $U = \sum_{i,t} Pr\{t\}u_i(t)$.

Definition 4.3. For any set S of profiles, let $Q(S) = \sum_{t \in S} Pr\{t\} \sum_{i,j} q'^j_i(t)$ and $U(S) = \sum_{t \in S} Pr\{t\} \sum_i u_i(t)$.

To make use of the Basic Formula, we partition the profiles that can possibly contribute to the Q term into three subsets S_0, S_1, S_2, and then use the IR-DIC Conditions to show that the U term (utility obtained by buyers) is greater than a certain linear combination of $Q(S_0), Q(S_1), Q(S_2)$. The Basic Formula then yields $r_D(\delta)$ as an upper bound to $M(\mathcal{F}_\delta)$.

Definition 4.4. Let $S_0 = \{(a, a)^n\}$ be the set containing a single element, namely, the lowest profile. Let S_1 be the set of 1-cheap profiles t satisfying $|I(t)| = 1$. Let S_2 be the set of 1-cheap profiles t satisfying $|I(t)| \ge 2$.

Fact 2. $q'^j_i(t) = 0$ for all $t \notin S_0 \cup S_1 \cup S_2$.

Recall that $p_0 = p^{2n}, p_1 = 2np^{2n-1}(1 - p), p_2 = 2p^n(1 - p^n - np^{n-1}(1 - p))$. They have the following interpretation as can be easily verified.

Fact 3. $Pr\{t \in S_\ell\} = p_\ell$ for $\ell \in \{0, 1, 2\}$, where t is distributed according to $\mathcal{F}_\delta$.

LEMMA 4.5.

$$Q(S_0) \le 2p_0, Q(S_1) \le p_1, Q(S_2) \le p_2. \tag{2}$$

PROOF. Any profile in S_1 or S_2 has exactly one cheap item, and the (only) profile in S_0 has two cheap items. Lemma 4.5 then follows from Fact 3. □

LEMMA 4.6.

$$M(\mathcal{F}_\delta) \le 2b(1 - p^n) + a \sum_{0 \le \ell \le 2} Q(S_\ell) - U. \tag{3}$$

PROOF. It follows from Fact 2 that $Q = Q(S_0) + Q(S_1) + Q(S_2)$. Lemma 4.6 then follows from the Basic Formula. □

Lemmas 4.5 and 4.6 set the stage. We are ready to invoke the incentive compatibility requirements to prove Theorem 4.1. This setting is also useful in the next section when we prove the BIC part of the Main Theorem.

4.1 Upper Bound to DIC Revenue

We prove Theorem 4.1 in this subsection. The key is to prove the following proposition.

Proposition 1. Any IR-DIC mechanism M must satisfy the following inequality:

$$U \geq (b - a)\Big(\frac{1 - p^2}{2p^2}Q(S_0) + \frac{1 - p}{2p}Q(S_1) + \frac{1 - p}{p}Q(S_2)\Big). \tag{4}$$

We first show that Theorem 4.1 follows from Proposition 1. It follows from Lemma 4.6 and Proposition 1 that, for any IR-DIC mechanism M, we have

$$M(\mathcal{F}_\delta) \leq 2(1 - p^n)b + Q(S_0)\Big[a - \frac{1 - p^2}{2p^2}(b - a)\Big]_+$$

$$+ Q(S_1)\Big[a - \frac{1 - p}{2p}(b - a)\Big]_+$$

$$+ Q(S_2)\Big[a - \frac{1 - p}{p}(b - a)\Big]_+.$$

With no negative terms, the above expression together with Lemma 4.5 immediately yield Theorem 4.1. Thus to establish Theorem 4.1, it suffices to prove Proposition 1.

Definition 4.7. For any $1 \leq i, i' \leq n$,
let $\tau_{i,i'}$ be the profile t such that $t_i^1 = t_{i'}^2 = b$ and all other $t_\ell^j = a$;
let $\tau_{i,0}$ be the profile t with $t_i^1 = b$ and all other $t_\ell^j = a$;
let $\tau_{0,i'}$ be the profile t with $t_{i'}^2 = b$ and all other $t_\ell^j = a$;
let $\tau_{0,0} = (a, a)^n$.

Fact 4. $S_0 = \{\tau_{0,0}\}$, $S_1 = \{\tau_{i,0}, \ \tau_{0,i} | \ 1 \leq i \leq n\}$.

Definition 4.8. For any $t \in S_2$ and $1 \leq i \leq n$, define $\tau_{t,i}$ as follows: let item j ($j \in \{1, 2\}$) be the cheap item for t; define $\tau_{t,i} = t'$ where $t'^j_i = b$ and $t'^{j'}_{i'} = t^{j'}_{i'}$ for all other $(i', j') \neq (i, j)$.

Definition 4.9. Let $S'_1 = \{\tau_{i,i'} | 1 \leq i, i' \leq n\}$. Let $S'_2 = \{\tau_{t,i} | t \in S_2, 1 \leq i \leq n\}$.

Fact 5. S'_1, S'_2 are disjoint sets of profiles containing no cheap items.
From Fact 5 and the IR Conditions, we have

$$U \geq U(S_1) + U(S'_1) + U(S'_2). \tag{5}$$

We now utilize the DIC-conditions to establish the following lemma relating the U and Q values on different types.

LEMMA 4.10.

$$U(S_1) \geq \frac{1 - p}{p}((b - a)Q(S_0) + 2U(S_0)), \tag{6}$$

$$U(S'_1) \geq \frac{1 - p}{2p}((b - a)Q(S_1) + U(S_1)), \tag{7}$$

$$U(S'_2) \geq \frac{1 - p}{p}((b - a)Q(S_2) + U(S_2)). \tag{8}$$

PROOF. The DIC-conditions require that, for all t_i, t'_i, t_{-i},

$$u_i(t_i, t_{-i}) \geq u_i(t'_i, t_{-i}) + \sum_j (t_i^j - t'^j_i)q_i^j(t'_i, t_{-i}).$$

We only need a subset of these conditions where $t_i > t'_i$. In such cases, we can use q'^j_i instead of q^j_i and write

DIC-Conditions: For all $t_i > t'_i$ and any t_{-i},

$$u_i(t_i, t_{-i}) \geq u_i(t'_i, t_{-i}) + \sum_j (t^j_i - t'^j_i) q'^j_i(t'_i, t_{-i}). \tag{9}$$

To prove Eq. 6, consider $t_i \in \{(b, a), (a, b)\}$, $t'_i = (a, a)$. We have

$$u_i(\tau_{i,0}) = u_i((b, a), (a, a)^{n-1}) \geq u_i(\tau_{0,0}) + (b - a) q'^1_i(\tau_{0,0}),$$

$$\text{and} \quad u_i(\tau_{0,i}) \geq u_i(\tau_{0,0}) + (b - a) q'^2_i(\tau_{0,0}). \tag{10}$$

By Fact 4 we have

$$U(S_1) = \sum_{t \in S_1} Pr\{t\} \sum_{i'} u_{i'}(t)$$

$$= p^{2n-1}(1 - p) \sum_i \Big(\sum_{i'} u_{i'}(\tau_{i,0}) + \sum_{i'} u_{i'}(\tau_{0,i}) \Big).$$

Using Eq. 10 and the IR Conditions $u_{i'}(t) \geq 0$, we obtain

$$U(S_1) \geq p^{2n-1}(1 - p) \Big(\sum_i u_i(\tau_{i,0}) + \sum_i u_i(\tau_{0,i}) \Big)$$

$$\geq p^{2n-1}(1 - p) \sum_i \big(2u_i(\tau_{0,0}) + (b - a)q'^1_i(\tau_{0,0}) + (b - a)q'^2_i(\tau_{0,0}) \big)$$

$$= \frac{1 - p}{p} Pr\{\tau_{0,0}\} \big(2 \sum_i u_i(\tau_{0,0}) + (b - a) \sum_j \sum_i q'^j_i(\tau_{0,0}) \big)$$

$$= \frac{1 - p}{p} (2U(S_0) + (b - a)Q(S_0)).$$

This proves Eq. 6, the first inequality in the Lemma.

We now prove Eq. 7. Write $S_1 = S^L_1 \cup S^R_1$ where $S^L_1 = \{\tau_{0,i} | 1 \leq i \leq n\}$ and $S^R_1 = \{\tau_{i,0} | 1 \leq i \leq n\}$. It suffices to prove for $x \in \{L, R\}$,

$$U(S'_1) \geq \frac{1 - p}{p} ((b - a)Q(S^x_1) + U(S^x_1)). \tag{11}$$

We prove Eq. 11 for $x = L$; the case for $x = R$ is similar.

$$(b - a)Q(S^L_1) + U(S^L_1) = \sum_{t \in S^L_1} Pr\{t\} \Big((b - a) \sum_j \sum_i q'^j_i(t) + \sum_i u_i(t) \Big)$$

$$= \sum_{t \in S^L_1} p^{2n-1}(1 - p) \sum_i \big((b - a)q'^1_i(t) + u_i(t) \big)$$

$$= p^{2n-1}(1 - p) \sum_{i'} \sum_i \big((b - a)q'^1_i(\tau_{0,i'}) + u_i(\tau_{0,i'}) \big). \tag{12}$$

Now consider the DIC-Conditions (Eq. 9) for $(t_i, t_{-i}) = \tau_{i,i'}$ and $(t'_i, t_{-i}) = \tau_{0,i'}$, which gives

$$u_i(\tau_{i,i'}) \geq u_i(\tau_{0,i'}) + (b - a)q'^1_i(\tau_{0,i'}). \tag{13}$$

From Eqs. 12 and 13, we obtain

$$(b-a)Q(S_1^L) + U(S_1^L) \le p^{2n-1}(1-p) \sum_i \sum_{i'} u_i(\tau_{i,i'})$$

$$\le \frac{p}{1-p} \sum_i \sum_{i'} Pr\{\tau_{i,i'}\} \sum_{i''} u_{i''}(\tau_{i,i'})$$

$$= \frac{p}{1-p} \sum_{t \in S_1'} Pr\{t\} \sum_i u_i(t)$$

$$= \frac{p}{1-p} U(S_1').$$

This proves Eq. 11, thus completing the proof of Eq. 7.

We now prove Eq. 8, the third inequality of Lemma 4.10. By definition

$$(b-a)Q(S_2) + U(S_2) = \sum_{t \in S_2} Pr\{t\}((b-a) \sum_{i,j} q'^j_i(t) + \sum_i u_i(t)). \tag{14}$$

Now observe that the DIC-Condition Eq. 9 for $\tau_{t,i} \in S_2'$ and $t \in S_2$ implies[2]

$$u_i(\tau_{t,i}) \ge u_i(t) + (b-a) \sum_j q'^j_i(t) \tag{15}$$

From Eq. 14 and 15, we obtain

$$(b-a)Q(S_2) + U(S_2) \le \sum_{t \in S_2} Pr\{t\} \sum_i u_i(\tau_{t,i})$$

$$= \frac{p}{1-p} \sum_{t \in S_2} \sum_i Pr\{\tau_{t,i}\} u_i(\tau_{t,i})$$

$$\le \frac{p}{1-p} \sum_{t \in S_2'} Pr\{t\} \sum_{i''} u_i(t)$$

$$= \frac{p}{1-p} U(S_2').$$

This proves Eq. 8. We have completed the proof of the Lemma 4.10. □

Proposition 1 can be straightforwardly derived from Lemma 4.10, Eq. 5, and the IR conditions $U(S_0), U(S_2) \ge 0$. This completes the proof of Proposition 1 and hence Theorem 4.1.

4.2 Realizing DIC Revenue

We turn to the proof of Theorem 4.2. We need to prove two statements.
Statement 1. $M_{D,\delta}$ is IR and DIC;
Statement 2. $M_{D,\delta}(\mathcal{F}_\delta) = r_D(\delta)$.

The proof of Statement 1 is straightforward by case analysis, and is omitted here (see the Appendix of arXiv: 1607.03685). For the rest of this subsection, we prove Statement 2. Here is the top level view of the proof. To show that the upper bound on revenue from Theorem 4.1 can be achieved, we demonstrate that several critical inequalities involved in the upper bound proof can be replaced by equalities. First, for mechanism $M_{D,\delta}$, it can be verified that Eqs. 3, 4 now are equalities, while

---

[2]If j is the cheap item in t, then $u_i(\tau_{t,i}) \ge u_i(t) + (b-a)q'^j_i(t)$. However, $q'^j_i(t) = \sum_{j'} q'^{j'}_i(t)$ in this case, since $q'^{j'}_i(t) = 0$ for $j' \ne j$.

Eq. 2 is replaced by $Q(S_0) = 2\alpha p_0, Q(S_1) = \beta p_1, Q(S_2) = \gamma p_2$. Combining these equalities gives us $M_{D,\delta}(\mathcal{F}_\delta) = r_D(\delta)$. We now give the details.

Fact 6. $u_i(t) = 0$ for all $t \notin S_1 \cup S_1' \cup S_2'$ and all i. Thus, $U = U(S_1) + U(S_1') + U(S_2')$.

Proof. From Eq. 1, we know that $u_i(t) \neq 0$ may occur only when $t = (t_i, t_{-i})$ and one of the following is valid: (a) $t_{-i} = (a, a)^{n-1}$ and $t_i \neq (a, a)$; (b) t_{-i} is 1-cheap and $t_i = (b, b)$. In case (a) we have $t \in S_1 \cup S_1'$, and in case (b) we have $t \in S_2'$. □

Fact 7.

$$\sum_{i,j} q'^j_i(t) = \begin{cases} 2\alpha & \text{if } t \in S_0 \\ \beta & \text{if } t \in S_1 \\ \gamma & \text{if } t \in S_2 \end{cases} \tag{16}$$

Proof. For the (only) profile t in S_0, the allocation function of $M_{D,\delta}$ specifies $\sum_{i,j} q'^j_i(t) = 2$ if $b < v_1$, and 0 otherwise. Similarly, for any profile $t \in S_1$, $\sum_{i,j} q'^j_i(t) = 1$ if $b < v_3$, and 0 otherwise; and for any profile $t \in S_2$, $\sum_{i,j} q'^j_i(t) = 1$ if $b < v_2$, and 0 otherwise. This is exactly the assertion of Fact 7. □

LEMMA 4.11.

$$Q(S_0) = 2p_0\alpha, Q(S_1) = p_1\beta, Q(S_2) = p_2\gamma. \tag{17}$$

PROOF. Follows immediately from Fact 3 and 7. □

LEMMA 4.12.

$$M_{D,\delta}(\mathcal{F}_\delta) = 2(1 - p^n)b + a \sum_{0 \leq \ell \leq 2} Q(S_\ell) - (U(S_1) + U(S_1') + U(S_2')).$$

PROOF. As under $M_{D,\delta}$ all the non-cheap items are allocated in full, the Basic Formula achieves equality, i.e. $M_{D,\delta}(\mathcal{F}_\delta) = 2(1 - p^n)b + Qa - U$. Also from Fact 2 we have $Q = Q(S_0) + Q(S_1) + Q(S_2)$, and from Fact 6 we have $U = U(S_1) + U(S_1') + U(S_2')$. Lemma 4.12 follows. □

LEMMA 4.13.

$$U(S_1) = (b - a)\frac{1 - p}{p}Q(S_0), \tag{18}$$

$$U(S_1') = (b - a)\frac{1}{2}((\frac{1 - p}{p})^2 Q(S_0) + (\frac{1 - p}{p})Q(S_1)), \tag{19}$$

$$U(S_2') = (b - a)\frac{1 - p}{p}Q(S_2). \tag{20}$$

PROOF. Eq. 18 can easily be derived from Eq. 1, Lemma 4.11, and Fact 3. We omit the proof here.

To prove Eq. 19, note that for any $\tau_{i,i'} \in S_1'$, Eq. 1 implies $\sum_{i''} u_{i''}(\tau_{i,i'}) = (\frac{\alpha}{n} + \beta)(b - a)$ if $i = i'$, and 0 otherwise. Thus we have

$$\begin{aligned} U(S_1') &= \sum_{t \in S_1'} Pr\{t\} \sum_{i''} u_{i''}(t) \\ &= \sum_{i,i'} Pr\{\tau_{i,i'}\} \sum_{i''} u_{i''}(\tau_{i,i'}) \\ &= \sum_i Pr\{\tau_{i,i}\}(\frac{\alpha}{n} + \beta)(b - a) \\ &= p^{2n-2}(1 - p)^2(\alpha + n\beta)(b - a). \end{aligned}$$

Making use of Lemma 4.11 and Fact 3, we obtain Eq. 19.

To prove Eq. 20, note that for any $t \in S_2, 1 \le i, i' \le n$ we have from Eq. 1

$$u_{i'}(\tau_{t,i}) = \begin{cases} \frac{\gamma}{|I(t)|}(b-a) & \text{if } i' = i \in I(t) \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

It follows that

$$
\begin{aligned}
U(S_2') &= \sum_{t' \in S_2'} Pr\{t'\} \sum_{i'} u_{i'}(t') \\
&= \sum_{t \in S_2} \sum_{i} \frac{1-p}{p} Pr\{t\} \sum_{i'} u_{i'}(\tau_{t,i}) \\
&= \frac{1-p}{p} \sum_{t \in S_2} Pr\{t\} \sum_{i \in I(t)} \frac{\gamma}{|I(t)|}(b-a) \\
&= \frac{1-p}{p} \gamma(b-a) \sum_{t \in S_2} Pr\{t\} \\
&= (b-a)\frac{1-p}{p} Q(S_2),
\end{aligned}
\tag{22}
$$

where we used Lemma 4.11 and Fact 3 in the last step. This proves Eq. 20. We have finished the proof of Lemma 4.13. □

Using Lemmas 4.11-4.13 and simplifying the above equation, we obtain

$$M_{D,\delta}(\mathcal{F}_\delta) = r_D(\delta).$$

This proves Statement 2, and completes the proof of Theorem 4.2.

5 BIC MAXIMUM REVENUE

In this section we give a proof of the Main Theorem for the Bayesian implementation:

THEOREM 5.1. *Any BIR-BIC mechanisms M must satisfy $M(\mathcal{F}_\delta) \le r_B(\delta)$.*

THEOREM 5.2. *$M_{B,\delta}$ is IR-BIC, and $M_{B,\delta}(\mathcal{F}_\delta) = r_B(\delta)$.*

The proofs of Theorem 5.1 and 5.2 follow the same top-level outline as the proofs of Theorem 4.1 and 4.2. Lemma 4.5 and 4.6 proved in Section 4 are valid for any mechanism M, and will also be the starting point for the BIC proof.

5.1 Upper Bound to BIC Revenue

We prove Theorem 5.1 in this subsection. The key is to prove the following proposition.

Proposition 2. Any BIR-BIC mechanism M must satisfy the following inequality:

$$U \ge (b-a)\left(\frac{1-p^2}{2p^2} Q(S_0) + \frac{1-p}{2p}(Q(S_1) + Q(S_2))\right).$$

Theorem 5.1 can be derived from Lemma 4.5, 4.6 and Proposition 2 in exactly the same way as Theorem 4.1's derivation from Lemma 4.5, 4.6 and Proposition 1, and will not be repeated here. It remains to prove Proposition 2.

We use a subset of the BIR-BIC Conditions in our proof; these conditions are listed below for easy reference.

(a) BIR Condition: For each i,

$$\bar{u}_i(t_i) \geq 0 \text{ where } t_i = (a, a). \tag{23}$$

(b) BIC Condition: For each i,

$$\bar{u}_i(b, a) \geq \bar{u}_i(a, a) + (b - a)\bar{q}'^{1}_i(a, a). \tag{24}$$

$$\bar{u}_i(a, b) \geq \bar{u}_i(a, a) + (b - a)\bar{q}'^{2}_i(a, a). \tag{25}$$

$$\bar{u}_i(b, b) \geq \bar{u}_i(a, b) + (b - a)\bar{q}'^{1}_i(a, b). \tag{26}$$

$$\bar{u}_i(b, b) \geq \bar{u}_i(b, a) + (b - a)\bar{q}'^{2}_i(b, a). \tag{27}$$

The plan is to use Eqs. 23-27 to obtain a lower bound on U in terms of $Q(S_0)$, $Q(S_1)$ and $Q(S_2)$.

LEMMA 5.3. *For each i,*

$$\bar{u}_i(b, a) + \bar{u}_i(a, b) \geq (b - a) \sum_j \bar{q}'^{j}_i(a, a).$$

PROOF. Immediate from Eqs. 23-25. □

LEMMA 5.4. *For each i,*

$$\bar{u}_i(b, b) \geq \frac{1}{2}(b - a) \sum_j \bar{q}'^{j}_i(a, a)$$

$$+ \frac{1}{2}(b - a) \sum_j \left(\bar{q}'^{j}_i(a, b) + \bar{q}'^{j}_i(b, a) \right).$$

PROOF. Adding up Eqs. 26 and 27, we obtain

$$\bar{u}_i(b, b) \geq \frac{1}{2} \left(\bar{u}_i(b, a) + \bar{u}_i(b, a) \right)$$

$$+ \frac{1}{2}(b - a) \sum_j \left(\bar{q}'^{j}_i(a, b) + \bar{q}'^{j}_i(b, a) \right), \tag{28}$$

where we have used the fact that $q'^{2}_i((a, b), t_{-i}) = q'^{1}_i((b, a), t_{-i}) = 0$ for all t_{-i}. Lemma 5.4 now follows by using Lemma 5.3 on Eq. 28. □

We now express U as a convex combination of the left-hand sides of Eq 23, Lemmas 5.3 and 5.4, and obtain a lower bound in terms of $Q(S_\ell)$:

$$U = \sum_i \sum_t Pr\{t\}u_i(t)$$

$$= p^2 \sum_i \bar{u}_i(a, a)$$

$$+ p(1 - p) \sum_i \left(\bar{u}_i(b, a) + \bar{u}_i(a, b) \right)$$

$$+ (1 - p)^2 \sum_i \bar{u}_i(b, b)$$

$$\geq C_1 + C_2, \tag{29}$$

where

$$C_1 = (b-a)\left(p(1-p) + \frac{1}{2}(1-p)^2\right)\left[\sum_i \sum_{t_{-i}} Pr\{t_{-i}\} \sum_j q'^j_i((a,a), t_{-i})\right]$$

$$= (b-a)\frac{1-p^2}{2}\sum_i \sum_{t_{-i}} Pr\{t_{-i}\} \sum_j q'^j_i((a,a), t_{-i}), \tag{30}$$

$$\text{and} \quad C_2 = \frac{1}{2}(b-a)(1-p)^2 \sum_i \sum_{t_{-i}} Pr\{t_{-i}\} \sum_j \left(q'^j_i((a,b), t_{-i}) + q'^j_i((b,a), t_{-i})\right). \tag{31}$$

Separating out the $t_{-i} = (a,a)^{n-1}$ term in Eq. 30, we obtain

$$C_1 = (b-a)\frac{1-p^2}{2}\left[\sum_i p^{2n-2}\sum_j q'^j_i(\tau_{00})\right.$$

$$\left.+ \sum_i \sum_{t_{-i}\neq(a,a)^{n-1}} Pr\{t_{-i}\} \sum_j q'^j_i((a,a), t_{-i})\right]$$

$$\geq (b-a)\frac{1-p^2}{2p^2}Pr\{\tau_{00}\}\sum_{i,j} q'^j_i(\tau_{00})$$

$$+ (b-a)\frac{1-p}{2p}\sum_{\substack{t,i,t_i=(a,a)\\ t_{-i}\neq(a,a)^{n-1}}} Pr\{t\} \sum_j q'^j_i(t),$$

$$C_2 = (b-a)\frac{1-p}{2p}\sum_{\substack{t,\ i\\ t_i\in\{(a,b),(b,a)\}}} Pr\{t\} \sum_j q'^j_i(t).$$

It follows that

$$C_1 + C_2 \geq (b-a)\frac{1-p^2}{2p^2}Q(S_0)$$

$$+ (b-a)\frac{1-p}{2p}\sum_{t\neq\tau_{00}}\sum_{i,t_i\neq(b,b)} Pr\{t\} \sum_j q'^j_i(t).$$

Now, noting that $\sum_j q'^j_i(t) = 0$ if $t_i = (b,b)$, we have

$$\sum_{t\neq\tau_{00}}\sum_{i,t_i\neq(b,b)} Pr\{t\} \sum_j q'^j_i(t) = \sum_{t\neq\tau_{00}} Pr\{t\} \sum_i \sum_j q'^j_i(t)$$

$$= \sum_{t\in S_1\cup S_2} Pr\{t\} \sum_i \sum_j q'^j_i(t)$$

$$= Q(S_1) + Q(S_2) \tag{32}$$

where we have used Fact 2.

It follows from Eqs. 30-32 that

$$U \geq (b-a)\frac{1-p^2}{2p^2}Q(S_0) + (b-a)\frac{1-p}{2p}(Q(S_1) + Q(S_2)).$$

This proves Proposition 2, and completes the proof of Theorem 5.1.

5.2 Realizing BIC Revenue

To prove Theorem 5.2, it suffices to prove the following two statements.

Statement 3. $M_{B,\delta}$ is IR and BIC.

Statement 4. $M_{B,\delta}(\mathcal{F}_\delta) = r_B(\delta)$.

The proof of Statement 3 is straightforward, and is omitted here (see the Appendix of arXiv: 1607.03685). The rest of this subsection is devoted to the proof of Statement 4. The statement is clearly true if $b \geq v_3$ (i.e. *Case 3* in the definition of $M_{B,\delta}$), since in this case by definition $r_B(\delta) = r_D(\delta)$, $M_B(\mathcal{F}_\delta) = M_D(\mathcal{F}_\delta)$, and Theorem 4.2 has established $M_D(\mathcal{F}_\delta) = r_D(\delta)$. Thus we can assume $b \in (a, v_3)$ (i.e. Case 1 or 2). Note that in this situation $\beta = 1$ and $\alpha \in \{0, 1\}$.

The proof follows essentially the same outline as the proof of Statement 2 in Section 4.2. Fact 2, 3, 6 remain true; Fact 7, Lemmas 4.11, 4.12 are modified to the following.

Fact 8.

$$\sum_{i,j} q'^j_i(t) = \begin{cases} 2\alpha & \text{if } t \in S_0 \\ \beta & \text{if } t \in S_1 \cup S_2. \end{cases}$$

LEMMA 5.5. $Q(S_0) = 2p_0\alpha, Q(S_1) = p_1\beta, \text{ and } Q(S_2) = p_2\beta.$

LEMMA 5.6. $M_{B,\delta}(\mathcal{F}_\delta) = 2b(1 - p^n) + a \sum_{0 \leq \ell \leq 2} Q(S_\ell) - (U(S_1) + U(S'_1) + U(S'_2)).$

The two lemmas above are straightforward to prove. Finally, Lemma 4.13 is modified to the following:

LEMMA 5.7.

$$U(S_1) = (b - a)\frac{1 - p}{p}Q(S_0), \tag{33}$$

$$U(S'_1) = (b - a)\frac{1}{2}((\frac{1 - p}{p})^2 Q(S_0) + (\frac{1 - p}{p})Q(S_1)), \tag{34}$$

$$U(S'_2) = (b - a)\frac{1 - p}{2p}Q(S_2). \tag{35}$$

The proof of Eqs. 33-34 is exactly the same as in the proof of Eqs. 18-19 in Lemma 4.13. The proof of Eq. 35 is also similar to the proof of Eq. 20 in Lemma 4.13. We omit the details here. Use Lemmas 5.5-5.7 and simplify, we obtain

$$M_{B,\delta}(\mathcal{F}_\delta) = r_B(\delta).$$

This proves Statement 4, and completes the proof of Theorem 5.2.

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